

# S-Matrix for AdS from General Boundary QFT

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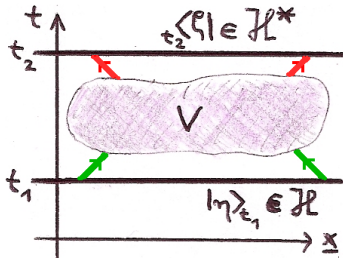
**Abstract.** The General Boundary Formulation (GBF) is a new framework for studying quantum theories. After concise overviews of the GBF and Schrödinger-Feynman quantization we apply the GBF to resolve a well known problem on Anti-deSitter spacetime where due to the lack of temporally asymptotic free states the usual S-matrix cannot be defined. We construct a different type of S-matrix plus propagators for free and interacting real Klein-Gordon theory.

## 1. Introduction

The General Boundary Formulation (GBF) is a new framework for studying quantum theories [1]. The respective theory is considered in some spacetime region, and the quantum states live in a Hilbert space associated to the region's boundary. Amplitudes and a probability interpretation can be assigned to the states. Not needing a fixed metric, the GBF is a promising framework for quantum theories treating the metric as a dynamical object. Using Schrödinger-Feynman quantization, in [2] the GBF reproduces the usual S-matrix for real Klein-Gordon theory on Minkowski spacetime, plus an equivalent S-matrix of different type. In [3] the same has been achieved for deSitter spacetime. The techniques used for both types of S-matrices are generalized for a wide class of spacetimes in [4] and now applied here for Anti-deSitter spacetime (AdS), where due to the lack of (temporal) asymptotically free states the usual S-matrix cannot be defined. The paper is structured as follows: After short reviews of the standard S-matrix (Section 2), AdS (3), the GBF (4) and Schrödinger-Feynman quantization (5), we apply in (6) the GBF for constructing the different type of S-matrix for real Klein-Gordon (KG) theory.

## 2. Standard construction of the S-matrix

In standard QFT there is *one* Hilbert space  $\mathcal{H}$  of free states. The S-matrix  $\mathcal{S} : \mathcal{H} \rightarrow \mathcal{H}$  is a unitary operator. Matrix elements are obtained as large-time limit  $\mathcal{S}_{\eta,\zeta} \sim \lim_{t \rightarrow \infty} \langle \zeta | \mathcal{U}_{[-t,t]} | \eta \rangle_{-t}$ ,



with  $\mathcal{U}_{[-t,t]}$  being the time-evolution operator.

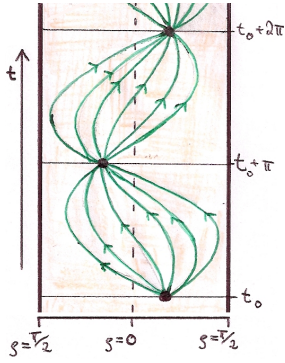
**Figure 2.1.** Slice of spacetime between the two equal-time hypersurfaces at  $t_1$  and  $t_2$ . Usually it is assumed that the interaction is "switched on" only between times  $t_1$  and  $t_2$ , then  $|\eta\rangle_{t_1}$  and  ${}_{t_2}\langle\zeta|$  are free states. Setting  $t_1 = -t$  and  $t_2 = +t$  and taking the limit  $t \rightarrow \infty$  the states become asymptotically free while the interaction can now be "switched on" on all of spacetime.

There is however a physically more convincing way to describe this. Physical interactions are not switched on and off by *time*, but decrease with the *distance* between interacting particles.

Consider interacting massive particles leaving the interaction region  $V$  in 2.1. When their separation becomes large, the state can be considered free and the particles as moving along timelike geodesics. For later times the states remain free iff spacetime geometry is such that the separation remains large, which in curved spacetime cannot be taken for granted, see figure 3.1.

### 3. Anti-deSitter spacetime (AdS)

We coordinatize (the universal covering version of) Lorentzian AdS of dimension  $(d+1)$  by  $x = (t, \rho, \Omega)$ , with time  $t \in (-\infty, +\infty)$ , radial coordinate  $\rho \in [0, \frac{\pi}{2})$ , and angles  $\Omega = (\theta_1, \dots, \theta_{d-1})$  such that  $\mathbb{S}^{d-1}$  is covered. The metric is  $ds_{\text{AdS}}^2 = R_{\text{AdS}}^2 \cos^{-2}\rho (dt^2 - d\rho^2 - \sin^2\rho ds_{\mathbb{S}^{d-1}}^2)$ , with the curvature radius  $R_{\text{AdS}}$ . For large  $R_{\text{AdS}}$  the AdS metric approximates the Minkowski metric. The free Lagrangian of real KG theory  $\mathcal{L} = \frac{1}{2}\sqrt{|g|} [g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2]$  yields as equation of motion the KG equation  $(\square + m^2)\phi = 0$ . Its solutions are the modes  $\phi_{\omega l \underline{m}}(x)$ , with frequency  $\omega \in (-\infty, +\infty)$  and angular quantum numbers  $(l, \underline{m})$  such that the spherical harmonics  $Y_l^{\underline{m}}(\Omega)$  form an orthonormal base (ONB) on  $\mathbb{S}^{d-1}$ . A separation ansatz [5] gives  $\phi_{\omega l \underline{m}}(x) = e^{-i\omega t} Y_l^{\underline{m}}(\Omega) S_{\omega, l}(\rho)$ . We consider only the case of odd  $d$ , where  $S_{\omega l}$  can be any linear combination of the two linear independent solutions of the radial part of the KG equation given by  ${}^1S_{\omega, l}(\rho) = \sin^l\rho \cos^{\tilde{m}+\rho} F(\alpha_+, \beta_+; \gamma^S; \sin^2\rho)$  and  ${}^2S_{\omega, l}^{\text{odd}}(\rho) = (\sin\rho)^{2-l-d} \cos^{\tilde{m}+\rho} F(\alpha_+ - \gamma^S + 1, \beta_+ - \gamma^S + 1; 2 - \gamma^S; \sin^2\rho)$ .



$F$  is the hypergeometric function with  $\alpha_+, \beta_+, \gamma^S$  depending on  $\omega, l$ , mass  $m$  and spatial dimension  $d$ .

**Figure 3.1.** Penrose diagram of AdS. Only temporal and radial directions are drawn. The negative curvature makes the timelike geodesics emanating from an arbitrary point at  $t_0$  reconverge in periods of  $\pi$ , see [6]. Thus there exists no time after which two particles leaving a scattering process are forever separated by large distances, i.e., after which interactions can be neglected forever. Therefore on AdS temporally asymptotic states do not exist and the standard S-matrix cannot be defined.

### 4. General Boundary Formulation (GBF)

To see how another type of S-matrix can be constructed for AdS now we outline the GBF [1]. To each hypersurface  $\Sigma$  (of codimension one) in a spacetime is associated its quantum state space  $\mathcal{H}_\Sigma$ . Each region  $M$  (of top dimension) has associated its boundary state space  $\mathcal{H}_{\partial M}$ , see fig. 4.1. Amplitudes for the quantum states are calculated with the amplitude map  $\rho_M : \mathcal{H}_{\partial M} \rightarrow \mathbb{C}$ .

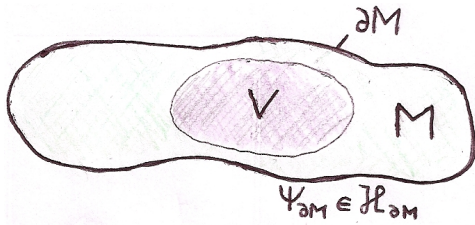


Figure 4.1.

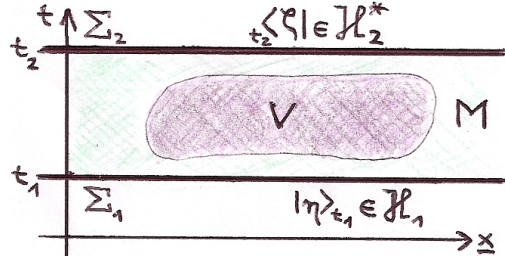
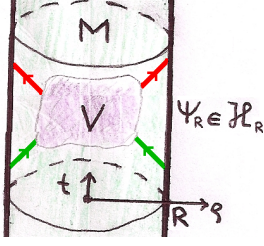


Figure 4.2.

Figure 4.2 shows how this general description includes the standard setting [1, 7]. There,  $M$  is the spacetime slice  $[t_1, t_2] \times \mathbb{R}^3$  with boundary  $\partial M = \Sigma_1 \sqcup \Sigma_2$ , a disjoint union of two hypersurfaces. The boundary state space is the tensor product  $\mathcal{H}_{\partial M} = \mathcal{H}_1 \otimes \mathcal{H}_2^*$ . The amplitude map  $\rho_M : \mathcal{H}_1 \otimes \mathcal{H}_2^* \rightarrow \mathbb{C}$  is now such that it is related to time evolution by  $\rho_M(|\eta\rangle_{t_1} \otimes |\zeta\rangle_{t_2}) = {}_2\langle \zeta | \mathcal{U}_{[t_1, t_2]} | \eta \rangle_1$ . The tensor product  $|\eta\rangle_{t_1} \otimes |\zeta\rangle_{t_2}$  also writes as *one* boundary state  $\psi_{\partial M} \in \mathcal{H}_{\partial M}$  (dropping bra-ket notation).



**Figure 4.3.** Hypercylinder region  $M = \mathbb{R} \times \mathbb{B}^d$ : a solid ball of radius  $R$  in space times all of time [2]. The boundary  $\partial M$  is the *single* hypersurface  $\Sigma_R := \mathbb{R} \times \mathbb{S}^{d-1}$ . An incoming (outgoing) particle is now one that enters (leaves)  $M$  at any time. The boundary state space decomposes  $\mathcal{H}_{\partial M} = \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$  into spaces of purely incoming and outgoing states. The amplitude  $\rho_M$  of *one* boundary state  $\psi_{\partial M} = \psi_{\text{in}} \otimes \psi_{\text{out}}$  is thus related to observing its outcoming part  $\psi_{\text{out}}$  given  $\psi_{\text{in}}$  was prepared.

If we assume spacetime geometry being such that large radius  $R$  implies large distance for two points on  $\Sigma_R$ , then for  $R \rightarrow \infty$  interactions can be neglected and the boundary state is (radially) asymptotically free. This is the case for AdS and so the problem of no (temporally) asymptotic states can be avoided. The large- $R$  limit of  $\rho_M(\psi_{\partial M})$  can thus be interpreted as a new type of S-matrix.

### 5. Schrödinger-Feynman quantization

In the Schrödinger representation [8] the states are complex function(al)s on field configurations on a hypersurface. Let  $K_\Sigma$  the space of field configurations on  $\Sigma$ , then  $\psi_\Sigma : K_\Sigma \rightarrow \mathbb{C}$ . Feynman quantization means that amplitudes are calculated as a path-integral with  $Z_M$  called field propagator:

$$\rho_M(\psi_{\partial M}) = \int_{K_{\partial M}} \mathcal{D}\varphi \psi_{\partial M}(\varphi) Z_M(\varphi) \quad Z_M(\varphi) = \int_{K_M, \phi|_{\partial M} = \varphi} \mathcal{D}\phi e^{iS_M(\phi)}.$$

The vacuum  $\psi_{R,0}$  for a hypercylinder  $\Sigma_R$  is a Gaussian [9, 10], with  $\hat{A}_R$  called vacuum operator:

$$\psi_{R,0}(\varphi) \sim \exp\left(-\frac{1}{2} \int_{\Sigma_R} dt d\Omega \varphi(x) (\hat{A}_{\Sigma_R} \varphi)(x)\right) \quad \hat{A}_R = -i \sqrt{|g_{\Sigma_R} g^{\rho\rho}|} \left. \frac{\partial_\rho \hat{\Upsilon}(\rho)}{\hat{\Upsilon}(\rho)} \right|_{\rho=R}.$$

An ONS on  $\Sigma_R$  is given by the functions  $u_{\omega l m}(t, \Omega) \sim e^{-i\omega t} Y_l^m(\Omega)$ , and  $\hat{\Upsilon}$  is defined by its eigenvalues:  $\hat{\Upsilon}(\rho) u_{\omega l m}(t, \Omega) = [c_{\omega l m}^a S_{\omega, l}^1(\rho) + c_{\omega l m}^b S_{\omega, l}^{2, \text{odd}}(\rho)] u_{\omega l m}(t, \Omega)$ . The vacuum is thus determined by fixing the coefficients  $c^{a,b}$  ( $a, b$  are labels, not indices). We shall choose  $c_{\omega l m}^a = (p R_{\text{AdS}})^l / (2l+1)!!$  and  $c_{\omega l m}^b = -i(2l-1)!! / (p R_{\text{AdS}})^{l+1}$  with  $p := \sqrt{|\omega^2 - m^2|}$ , because in the limit  $R_{\text{AdS}} \rightarrow \infty$  this choice yields the well known Minkowski vacuum.

Next we can define coherent states. A coherent state is determined by its complex characterising function  $\eta$  and usually written with creators  $\hat{a}_{\omega l m}^\dagger$  as  $\psi_{R, \eta}(\varphi) \sim [\exp \int d\omega \sum_{l m} \eta_{\omega l m} \hat{a}_{\omega l m}^\dagger] \psi_{R,0}(\varphi)$ . In the interaction picture it can equivalently be written as  $\psi_{R, \xi}(\varphi) \sim [\exp \int dt d\Omega \xi(t, \Omega) \hat{\Upsilon}^{-1}(\rho) \varphi(t, \Omega)] \psi_{R,0}(\varphi)$ , see [4].

### 6. Radial S-matrix for AdS

Labeling all quantities of the free theory by a zero, calculating the free amplitude results in

$$\rho_{R,0}(\psi_{R, \xi}) = \int \mathcal{D}\varphi \psi_{R, \xi}(\varphi) Z_{R,0}(\varphi) = \exp \left[ \frac{1}{2} \int_{\Sigma_R} dt d\Omega \xi(t, \Omega) \hat{\mathcal{B}} \left( \frac{\bar{c}_b}{c_b} \xi(t, \Omega) + \overline{\xi(t, \Omega)} \right) \right],$$

with  $\hat{\mathcal{B}}$  defined by the eigenvalues  $p / (2R_{\text{AdS}}^{d-2}) \cdot (2l+1) / (2l+d-2)$  when acting on the  $u_{\omega l m}(t, \Omega)$ . Since the free amplitude is independent of  $R$  (not  $R_{\text{AdS}}$ ), taking the large- $R$  limit is easy and the amplitude can then be interpreted as radial S-matrix. Now we add a source field  $\mu$  to include interactions:  $S_{R, \mu}(\phi) = S_{R,0}(\phi) + \int_{\rho < R} d^{d+1}x \sqrt{|g|} \mu(x) \phi(x)$ . At this point we assume the source field to vanish for  $\rho \geq R$ . The resulting amplitude for a coherent state is

$$\rho_{R, \mu}(\psi_{R, \xi}) = \rho_{R,0}(\psi_{R, \xi}) \left[ \exp \int_{\rho < R} d^{d+1}x \sqrt{|g|} \mu \hat{\mathcal{C}} \hat{X}_a \xi \right] \left[ \exp \frac{i}{2} \int_{\rho < R} d^{d+1}x d^{d+1}x' \sqrt{|g(x)g(x')|} \mu(x) G_F(x, x') \mu(x') \right].$$

The operators are again defined by their eigenvalues: for  $\hat{X}_a(\rho)$  by  ${}^1S_{\omega,l}(\rho)$  and for  $\hat{\mathcal{C}}$  by  $i(pR_{\text{AdS}})^{l+1}[R_{\text{AdS}}^{d-1}(2l+d-2)(2l-1)!!]^{-1}$ . The Feynman propagator  $G_F$  fulfills the inhomogeneous KG equation  $(\square_{\text{AdS}} + m^2) G_F(x, x') = \delta^{(d)}(x-x')$  and is given by

$$G_F(x, x') = \int d\omega \sum_{l, \underline{m}} \left\{ \theta(\rho - \rho') \left( \hat{\mathcal{C}} \hat{X}_a(\rho') \overline{u_{\omega l \underline{m}}} \right)(t', \Omega') \left( \hat{\Upsilon}(\rho) u_{\omega l \underline{m}} \right)(t, \Omega) + (x \leftrightarrow x') \right\}.$$

The amplitude is thus independent of  $R$ , so we can take the limit  $R \rightarrow \infty$  (such that the source field can have support on the whole spacetime) and consider the amplitude as an S-matrix. For a field interaction with potential  $V$  and action  $S_{R,V}(\phi) = S_{R,0}(\phi) + \int_{\rho < R} d^{d+1}x \sqrt{|g|} V(x, \phi)$  we can use the functional derivative technique to obtain

$$\exp i S_{R,V}(\phi) = \left[ \exp i \int_{\rho < R} d^{d+1}x \sqrt{|g(x)|} \hat{V}\left(x, -i \frac{\delta}{\delta \mu(x)}\right) \right] \exp i S_{R,\mu}(\phi) \Big|_{\mu=0}$$

leading to the  $R$ -independent amplitude below whose large- $R$  limit is an S-matrix:

$$\rho_{R,V}(\psi) = \left[ \exp i \int_{\rho < R} d^4x \sqrt{|g(x)|} \hat{V}\left(x, -i \frac{\delta}{\delta \mu(x)}\right) \right] \rho_{R,\mu}(\psi) \Big|_{\mu=0}.$$

## 7. Conclusions

Using the GBF, an S-matrix for AdS can be constructed by calculating (generalized) amplitudes for the hypercylinder region. This circumvents the problem of not having available (temporally) asymptotic states on AdS, needed for the S-matrix of standard QFT. Future projects consist in applying holomorphic quantization [11, 12, 13] within the GBF framework, studying properties of this S-matrix and connections with the AdS/CFT correspondence [14], and relating our S-matrix to the one found by Giddings [15].

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